

Kuhn-Tucker conditions for a convex programming problem in Banach spaces partially ordered by cone with empty interior.

Feyzullah Ahmetoğlu

Faculty of Education, Giresun University,
Giresun, Turkey. e-mail: feyzullah.ahmetoglu@giresun.edu.tr

1. Introduction

During recent decades the theory of mathematical programming in infinite dimensional spaces has been studied extensively [1]-[7].

In order to obtain Kuhn-Tucker condition in mathematical programming, problems usually are formulated in spaces where a cone defining partial order has a nonempty interior. In these spaces the existence of a saddle point of the Lagrange function or Kuhn-Tucker conditions are established by using of some natural conditions like Slayter, regularity, etc. These well known methods fail in the cases when the cones defining partial order in the space have no interior points. $L_p[0, T]$ and $l_p(1 < p < \infty)$ spaces constitute examples for these cases. In the present paper we explore spaces not necessarily having nonempty interior of the cone defining partial order. We obtain a differential form of Kuhn-Tucker conditions for a convex programming problem in Banach spaces without strong restriction assuming the existence of nonempty interior of the cone defining partial order in the space.

2. Formulation of results.

Let X and Y be reflexive Banach spaces partially ordered by convex closed cones K and P , respectively. A linear bounded operator mapping X into Y we denote by A .

We investigate the problem of minimization of the continuously differentiable convex functional $I(x)$ under following additional constraints:

$$Ax \leq b \quad (b - Ax) \in P$$

$$x \geq 0 \quad (x \in K)$$

The problem can be shortly formulated as

$$I(x) \rightarrow \min \tag{1}$$

$$Ax \leq b \quad x \geq 0 \tag{2}$$

Definition 1 . We say that constraints (2) satisfy the strong simultaneity condition, if there exists $\epsilon_0 > 0$ such that for every $\bar{b} \in \{\bar{b} : \|\bar{b} - b\| \leq \epsilon_0\}$ the system $Ax \leq \bar{b}, x \geq 0$ has a solution.

A point $p \in M$ is called an internal point of M , if for each $z \in Y$ there exists a real number $\epsilon > 0$ such that for each λ satisfying $|\lambda| \leq \epsilon$ we have $p + \lambda z \in M$.

Lemma 1. Suppose that the constraints (2) satisfy the strong simultaneity condition. Then the set

$$M = \{z \in Y : b - Ax \geq z, x \geq 0\}$$

has internal points.

Proof. In order to prove the lemma it suffices to show that a zero point is an internal point of M . In other words, for each point $z \in Y, z \neq 0$, there exists a real number λ' , such that the constraints

$$b - Ax \geq \lambda z, \quad x \geq 0$$

are consistent for all $\lambda \in (0, \lambda')$. We choose $\lambda' = \frac{\epsilon_0}{\|z\|}$. Then for each $\lambda \in (0, \lambda')$ we have $\lambda\|z\| < \epsilon_0$. Since the conditions (2) are strongly simultaneous, $b - Ax \geq \lambda z, x \geq 0$ for each $\lambda \in (0, \lambda')$. The proof is completed.

Lemma 2. Suppose that the constraints (2) satisfy the strong simultaneity condition. Then the set

$$S = \{(z, p) \in Y \times R : b - Ax \geq z, I(x) \leq p, x \geq 0\}$$

has internal points.

Proof. Clearly, there exists x_0 such that

$$b - Ax_0 \geq 0 \tag{3}$$

We show that $(0, I(x) + 1)$ is an internal point of S . Let $\rho_0 = I(x^0) + 1$. Let us show that for each $(z, \rho) \in Y \times R$, there exists $\bar{\lambda}$, such that for arbitrary $\lambda \in (0, \bar{\lambda})$ we have $(\lambda z, \rho_0 + \lambda \rho) \in S$. In other words, for arbitrary $\lambda \in (0, \bar{\lambda})$ there exists $x_\lambda \geq 0$, such that $b - Ax_\lambda \geq \lambda z$ and $\rho_0 + \lambda \rho \geq I(x_\lambda)$.

By Lemma 1 $0 \in Z$ is an internal point of M . Therefore, there exist a real number $\lambda_0 > 0$ and a point \bar{x}^0 such that

$$b - A\bar{x}_0 \geq \lambda_0 z \quad (4)$$

By multiplying both sides of (3) by $1 - \frac{\lambda}{\lambda_0}$ and both sides of (4) by $\frac{\lambda}{\lambda_0}$ and taking their sum, we get

$$b - A\left(\frac{\lambda}{\lambda_0}\bar{x}_0 + \left(1 - \frac{\lambda}{\lambda_0}\right)x_0\right) \geq \lambda z$$

Let $x_\lambda = \frac{\lambda}{\lambda_0}\bar{x}_0 + \left(1 - \frac{\lambda}{\lambda_0}\right)x_0$. Then

$$b - Ax_\lambda \geq \lambda z \quad (5)$$

Since $I(x)$ is a convex functional we get

$$I(x_\lambda) \leq \left(1 - \frac{\lambda}{\lambda_0}\right)I(x^0) + \frac{\lambda}{\lambda_0}I(\bar{x}^0) = I(x_0) + \frac{\lambda}{\lambda_0}(I(\bar{x}_0) - I(x^0))$$

In order to prove $I(x_\lambda) \leq \rho_0 + \lambda \rho$ it is enough to establish the following inequality

$$I(x^0) + \frac{\lambda}{\lambda_0}(I(\bar{x}_0) - I(x^0)) \leq I(x^0) + 1 + \lambda \rho$$

The last inequality is held for all $\lambda\beta \leq 1$, where $\beta = \left|\frac{I(\bar{x}_0) - I(x^0)}{\lambda_0} - \rho\right|$. Thus, we can complete the proof by choosing $\bar{\lambda}$

$$\bar{\lambda} = \begin{cases} \min\{\lambda_0, 1/\beta\} & \text{if } \beta \neq 0 \\ \lambda_0 & \text{if } \beta = 0 \end{cases}$$

Lemma 2 is proved.

Let X^* and Y^* be the conjugate spaces of X and Y , respectively. The conjugate cone of K is K^* :

$$K^* = \{x^* \in X^* : (x^*, x) \geq 0 \text{ for all } x \in K\}$$

The conjugate cone of P is defined similarly.

Let X^* and Y^* be partially ordered by K^* and P^* , respectively.

Lemma 3. Suppose that the constraints (2) satisfy the strong simultaneity condition. Then for any $z^* \in P^*$, $z^* \neq 0$, there exists a point $x_{z^*} \geq 0$ such that

$$(z^*, b - Ax_{z^*}) > 0$$

Proof. For strong simultaneity of (2) for each $\xi \in Y$, $\|\xi\| \leq 1$, there exists a point $x_\xi \geq 0$ such that

$$b - Ax_\xi \geq \epsilon_0 \xi$$

Let $z^* \in P^*$ and $z^* \neq 0$. Obviously, there exists $z_0 \in Y$ such that

$$\sup_{\|z\| \leq 1} (z^*, z) = (z^*, z_0)$$

Since $\|z_0\| \leq 1$, there exists $x_{z^*} \geq 0$ satisfying

$$b - Ax_{z^*} \geq \epsilon_0 z_0$$

Now

$$(z^*, b - Ax_{z^*}) \geq \epsilon_0 (z^*, z_0) = \epsilon_0 \sup_{\|z\| \leq 1} (z^*, z) = \epsilon_0 \|z^*\| > 0$$

The lemma is proved.

The functional $L(x, z^*) = I(x) + (z^*, Ax - b)$ is called a Lagrange function.

Definition 2 . A pair $\langle x_0, z_0^* \rangle$ is said to be a saddle point of Lagrange function if $x_0 \geq 0$, $z_0^* \geq 0$ and for each $x \geq 0$, $z^* \geq 0$

$$L(x_0, z^*) \leq L(x_0, z_0^*) \leq L(x, z_0^*) \quad (6)$$

It can be easily shown that the existence of a saddle point of Lagrange function implies the existence of a solution of problem (1),(2). The inverse of this statement is also true:

Theorem 1. Suppose that the constraints (2) satisfy the strong simultaneity condition and the problem (1),(2) has a solution x_0 . Then there exists a non-zero linear functional z_0^* such that the pair $\langle x_0, z_0^* \rangle$ is a saddle point of Lagrange function.

Proof. By Lemma 2 the set

$$S = \{(z, \rho) \in Y \times R : b - Ax \geq z, I(x) \leq \rho, x \geq 0\}$$

has internal points. By Lemma 3 for each $z^* \in P^*$, $z^* \neq 0$, there exists a point x_{z^*} such that $(z^*, b - Ax_{z^*}) > 0$. Thus, the strong simultaneity condition implies both conditions of Theorem 1 of [1], which states the existence of a saddle point.

Let us prove the existence of a saddle point in our case. Consider the following sets in $Y \times R$

$$N = \{(z, \rho) \in Y \times R : z \geq 0, \rho \leq I(x_0)\}$$

$$N_1 = \{(z, \rho) \in Y \times R : z \geq 0, \rho < I(x_0)\}$$

The sets S, N and N_1 are convex sets. Let us show that $S \cap N_1 = \emptyset$. Indeed, if $x \geq 0$ and $Ax \leq b$, then for all $(z, \rho) \in S$ we have $\rho \geq I(x) \geq I(x_0)$. On the other hand, in N_1 $\rho < I(x_0)$. If $x \geq 0$ and $b - Ax \notin P$, then in N_1 $z \geq 0$ but in S it is not held. Done.

By Lemma 2, S has an internal point. As a result, S and N_1 are disjoint convex sets and S has an internal point. Therefore, by well-known separation theorem [2], there exist $(y_0^*, \rho_0) \in Y^* \times R$, $(y_0^*, \rho_0) \neq 0$ such that

$$\rho_0 \rho + (y_0^*, z) \geq \rho_0 r + (y_0^*, y) \quad (7)$$

for all $(z, \rho) \in S$ and $(y, r) \in N_1$.

Since the closure of N_1 is N , (7) is also held for all $(y, r) \in N$, which implies that $\rho_0 \geq 0$. Indeed, N_1 contains pairs with arbitrary small negative values of r . Therefore, if $\rho_0 < 0$ we can increase the right side of (7) as much as we wish and get a contradiction with (7).

Clearly, $(0, I(x_0)) \in S$. Thus, for each $z \leq 0$ we have $(z, I(x_0)) \in S$. On the other hand $(0, I(x_0)) \in N$. Then for each $z \leq 0$ by (7)

$$\rho_0 I(x_0) + (y_0^*, z) \geq \rho_0 I(x_0)$$

Consequently, for all $z \leq 0$ we get $(y_0^*, z) \geq 0$. Therefore, $y_0^* \leq 0$.

For each $x \geq 0$ we have $(b - Ax, I(x)) \in S$. Then from (7) we get

$$\rho_0 I(x) + (y_0^*, b - Ax) \geq \rho_0 I(x_0) \quad (8)$$

for each $x \geq 0$.

Let us show that $\rho_0 > 0$. Indeed, if $\rho_0 = 0$, then from (8) we get

$$(-y_0^*, b - Ax) \leq 0 \quad (9)$$

for each $x \geq 0$.

Since (2) are strong simultaneous (9) contradicts Lemma 3.

Thus, $\rho_0 > 0$ and $y_0^* \leq 0$. Let $z_0^* = -\frac{y_0^*}{\rho_0}$. Then $z_0^* \geq 0$ and from (8) we have

$$I(x) + (z_0^*, Ax - b) \geq I(x_0) \quad (10)$$

for each $x \geq 0$.

If we put $x = x_0$ in (10) we get

$$(z_0^*, Ax_0 - b) \geq 0$$

On the other hand $z_0^* \geq 0$, $Ax_0 \leq b$ and consequently $(z_0^*, Ax_0 - b) \leq 0$. Last two inequalities imply that

$$(z_0^*, Ax - b) = 0 \quad (11)$$

Now (10) implies the second inequality in (6).

Let us prove the first inequality. Clearly, $(z^*, Ax_0 - b) \leq 0$ for each $z^* \geq 0$. By using (11) we get

$$(z^*, Ax_0 - b) \leq (z_0^*, Ax_0 - b)$$

for each $z^* \geq 0$. Therefore,

$$I(x_0) + (z^*, Ax_0 - b) \leq I(x_0) + (z_0^*, Ax_0 - b)$$

for each $z^* \geq 0$. The first inequality of (6) is proved.

Now we state a theorem establishing the Kuhn - Tucker condition for the problem (1),(2).

Theorem 2. Suppose that the constraints (2) satisfy the strong simultaneity condition. Then the necessary and sufficient condition for the existence of a solution x_0 of the problem (1),(2) is the existence of a nonzero linear functional $z_0^* \geq 0$ such that the following conditions are held:

$$I'(x_0) + A^* z_0^* \geq 0 \quad (12)$$

$$(I'(x_0) + A^* z_0^*, x_0) = 0 \quad (13)$$

$$Ax - b \geq 0, x_0 \geq 0 \quad (14)$$

$$(z_0^*, Ax - b) = 0 \quad (15)$$

where $I'(x)$ is a gradient of $I(x)$, A^* is the operator adjoint to A .

Proof. Due to Theorem 1, in order to prove theorem we have to establish that the condition (6) is equivalent to the conditions (12)-(15).

Suppose that (6) is held. The second inequality of (6) means that x_0 is a minimal point of convex functional $L(x, z_0^*)$. By the convex differentiability of a linear functional for each $x \geq 0$

$$(L'_x(x_0, z_0^*), x - x_0) \geq 0$$

Since $L'_x(x_0, z_0^*) = I'(x_0) + A^*z_0^*$ we obtain that for each $x \geq 0$

$$(I'(x_0) + A^*z_0^*, x - x_0) \geq 0 \quad (16)$$

Consequently, $I'(x_0) + A^*z_0^* \geq 0$.

Put $x = 0$ in (16):

$$I'(x_0) + A^*z_0^*, x_0 \leq 0$$

On the other hand, $I'(x_0) + A^*z_0^*, x_0 \geq 0$. Last two inequalities imply (13).

First inequality of (6) implies that for each $z^* \geq 0$

$$(z^*, Ax_0 - b) \leq (z_0^*, Ax_0 - b) \quad (17)$$

and consequently, for each $z^* \geq 0$

$$(z^*, Ax_0 - b) \leq 0 \quad (18)$$

or $Ax_0 - b \leq 0$.

Now we get $(z_0^*, Ax_0 - b) \geq 0$ by putting $z^* = 0$ in (17). On the other hand, $z_0^* \geq 0$, $Ax_0 - b \leq 0$ and hence $(z_0^*, Ax_0 - b) \leq 0$. Last two inequalities imply (15).

Now suppose that (12)-(15) are held. From (12) we get that for all $x \geq 0$

$$(I'(x_0) + A^*z_0^*, x) \geq 0$$

Now by using (13) we get that for all $x \geq 0$

$$(I'(x_0) + A^*z_0^*, x - x_0) \geq 0$$

In other words, for all $x \geq 0$

$$(L'_x(x_0, z_0^*), x - x_0) \geq 0$$

The last inequality is a necessary and sufficient condition for x_0 to be a minimal point of $L(x, z_0^*)$ for $x \geq 0$. Therefore, for all $x \geq 0$ we get $L(x_0, z_0^*) \leq L(x, z_0^*)$. Thus, the right side of (6) is proved.

From (14) we get $(z^*, Ax_0 - b) \leq 0$ for all $z^* \geq 0$. Now by (15), we get $(z^*, Ax_0 - b) \leq (z_0^*, Ax_0 - b)$ for all $z^* \geq 0$. Therefore, $L(x_0, z^*) \leq L(x_0, z_0^*)$ for all $z^* \geq 0$. Thus, the left side of (6) also is proved.

Remark. It can be readily shown that the strong simultaneity condition (2) is equivalent to the following condition

$$0 \in \text{int}(AK + b + P)$$

Clearly, $AK + b + P$ can have interior points even if P has no interior points. It means that the strong simultaneity condition can be held in cases when Slater condition is not held.

Proposition. In the case when $\text{int}P \neq \emptyset$, the Slater and the strong simultaneity conditions are equivalent.

Proof. Suppose that the Slater condition is held: there is a point $x_0 \geq 0$ such that $b - Ax_0 \in \text{int}P$. Then readily the strong simultaneity condition is held.

Now let the strong simultaneity condition is held. Then there exists a real number $\rho > 0$ such that for each $y \in S_\rho$ (S_ρ is a sphere with radius ρ centered at 0) $b - Ax \geq y$, $x \geq 0$. Clearly, the strong simultaneity condition can be written as

$$S_\rho \subset AK - b + P \quad (19)$$

In order to prove that the Slater condition is held we show that there exists a point $x_0 \geq 0$ and a real number $\rho_1 > 0$ such that

$$S_{\rho_1} \subset Ax_0 - b + P$$

It suffices to show that

$$\text{int}P \cap b - AK \neq \emptyset$$

Suppose the contrary: $\text{int}P \cap b - AK = \emptyset$. Since P and $b - AK$ are convex, by separation theorem [2] there exists a linear functional $z_0^* \in Y^*$, $z_0^* \neq 0$ such that

$$(z_0^*, P) \leq 0 \leq (z_0^*, b - AK)$$

or equivalently, $(z_0^*, AK - b + P) \leq 0$. From (16) we get $(z_0^*, S_\rho) \leq 0$. Thus, $z_0^* = 0$. This is a contradiction. The proof is completed.

References

- [1] Hurwitz L. , Udzava H. Stanford University Press (1958)

- [2] Danford N., Schwartz J.T. Linear Operators, 1, (1988)
- [3] Burachik R.S., Jeyakumar V. Mathematical Programming, Springer,104,2-3 (2005).
- [4] Evans L.C., Gomez D. Control, Optimization and Calculus of Variations, v.8, (2002).
- [5] Molanowski K. , Journal of Applied mathematics and Optimization, 25, 1, (1992).
- [6] Chen S.Y., Wu S.Y., Journal of Computational and Applied Mathematics, 213, 2, (2008).
- [7] Ito S. Journal of Industrial and Management Optimization, v.6, 1, (2010).